MA 410, Fall 2011

Assignment 10

This homework is due *Wednesday*, December 7th.

There are total 38 points in this assignment. 34 points is considered 100%. If you go over 34 points, you will get over 100% for this homework (up to 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper. Your solutions should contain full proofs/calculations (unless stated otherwise in the problem). Bare answers will not earn you much.

This assignment covers sections 7.1, 7.3, 7.4 in O'Neill.

(1) (Part of 7.1.1) Let M be a region of $\mathbb{R}^2$ with inner product $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v} \bullet \mathbf{v}/h^2$, $h > 0$. Show that:

  (a) [2pt] Angle between vectors $\mathbf{v}, \mathbf{w}$ in the sense of $\langle \cdot, \cdot \rangle$ coincides with the angle between them in the sense of Euclidean dot product $\bullet$. (Note: hence, the terminology: such geometric surface is called *conformal* with *ruler function h*.)

  (b) [2pt] The speed of curve $\alpha = (\alpha_1, \alpha_2)$ is $\sqrt{{\alpha_1'}^2 + {\alpha_2'}^2}/h(\alpha)$.

  (c) [2pt] hU_1, hU_2 is a frame field with dual 1-forms du/h, dv/h.

(2) (Part of 7.1.2, 7.3.1) The *Poincaré half-plane* is the upper half plane $v > 0$ in (u, v)-plane $\mathbb{R}^2$ with metric $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v} \bullet \mathbf{w}/v^2$. If α is a curve $(r \cos t, r \sin t)$, $0 < t < \pi$, with constant $r > 0$,

  (a) [3pt] show the speed of α is $\csc t$. (Note: in particular, does not depend on r.)

  (b) [2pt] Deduce that although the Euclidean length of α is πr, its Poincaré length is infinite.

  (c) [3pt] Show that the connection form of the frame field $E_1 = vU_1, E_2 = vU_2$ is $\omega_{12} = du/v = \theta_1$.

  (d) [3pt] Express velocity and acceleration of the curve α in terms of E_1, E_2 (α as defined above).

  (e) [3pt] Express velocity and acceleration of for the Euclidean straight line $\beta(t) = (ct, st)$ in terms of E_1, E_2, (c, s are constants such that $c^2 + s^2 = 1$, and values of t are such that $st > 0$). Verify calculations by checking that $\langle \beta', \beta' \rangle' = 2\langle \beta', \beta'' \rangle$.

(3) (7.1.4) (*Coordinate definition of a metric.*)

  (a) [4pt] If a, b, c are numbers such that $a > 0, c > 0, ac - b^2 > 0$, then the formula

$$\langle \mathbf{v}, \mathbf{w} \rangle = av_1w_1 + b(v_1w_2 + v_2w_1) + cv_2w_2$$

  defines an inner product on $\mathbb{R}^2$. (*Hint:* To prove positive definiteness, consider $(v_1\sqrt{a} \pm v_2\sqrt{c})^2 \geq 0$ and compare it to the required inequality.)

— *see next page* —

(b) [4pt] Let $\mathbf{x} : D \to M$ be a coordinate patch in an abstract (generally speaking, not geometric) surface M. Given differentiable functions $E, F, G : D \to \mathbb{R}$ such that

$$E > 0, \quad G > 0, \quad EG > F^2,$$

prove that there is a unique metric $\langle \cdot, \cdot \rangle$ on the image of $\mathbf{x}$ such that

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle, \quad E = \langle \mathbf{x}_v, \mathbf{x}_v \rangle.$$

(4) [3pt] (7.4.2) If $\gamma_{\mathbf{v}}$ is the unique geodesic in M with initial velocity $\mathbf{v}$, show that for any number c, $\gamma_{c\mathbf{v}} = \gamma_{\mathbf{v}}(ct)$ for all t.

(5) (7.4.6) In the projective plane $P(r)$ of radius r (sphere $\Sigma(r)$ of radius r with identified antipodal points), prove:

(a) [3pt] The geodesics are simple (non self-intersecting) closed curves of length πr.

(b) [2pt] There is a unique geodesic route through any two distinct point.

(c) [2pt] Two distinct geodesic routes meet in exactly one point.

(*Hint:* Within "small" patch (here, "small" means that image of patch fits inside a hemisphere), geometries of sphere $\Sigma(r)$ and $P(r)$ are the same.)